Inverse Trigonometric Functions

MATHEMATICS

Board

INVERSE FUNCTION

Let f be a function defined from a set A to a set B, i.e. f : A → B and g be a function defined from the set B to the set A, i.e., g : B → A; then the function g is said to be inverse of f if

$g\{f(x)\} = x, \forall x \in A$ and the function g is denoted by f^{-1}.

Properties of inverse of a function :

(i) The inverse of bijection is unique.

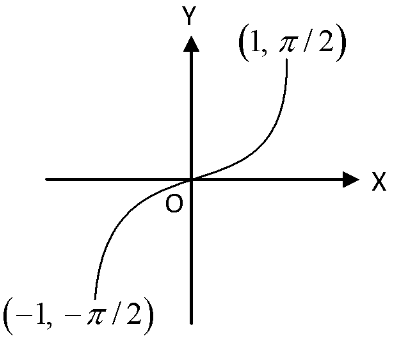
(ii) If f : A → B is bijection and g : B → A is inverse of f, then

$$f \circ g = I_B \text{ and } g \circ f = I_A$$

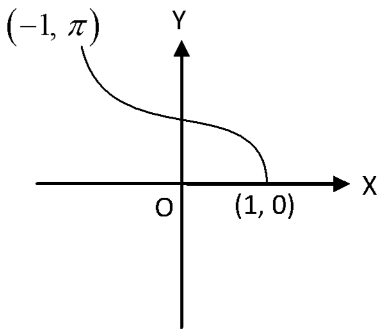
where, I_A and I_B are identity functions on the sets A and B respectively.

Graphs of inverse trigonometric functions

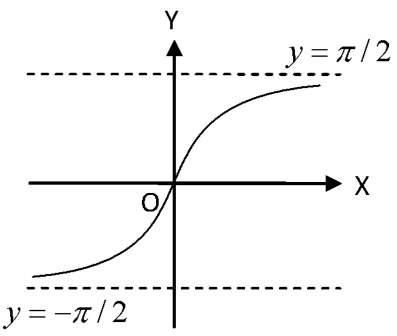
(i) Graph of $y = \sin^{-1}x$



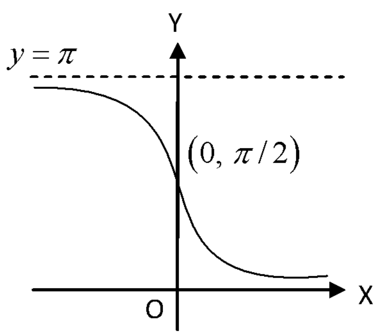
(ii) Graph of $y = \cos^{-1}x$



(iii) Graph of $y = \tan^{-1}x$

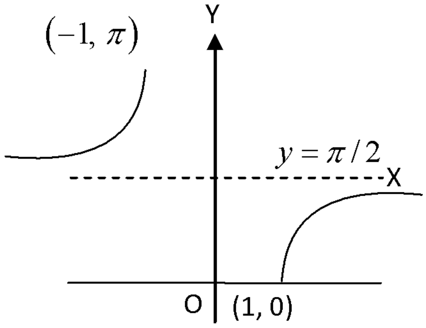


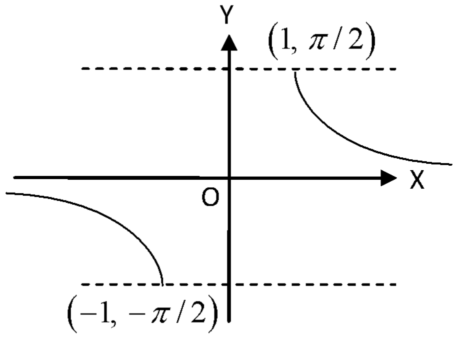
(iv) Graph of $y = \cot^{-1} x$



(v) Graph of $y = \sec^{-1} x$

(v) Graph of $y = \text{cosec}^{-1}x$





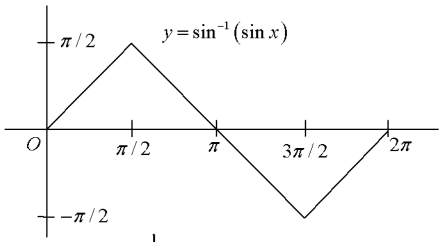
Domain and range of Inverse Trigonometric Functions

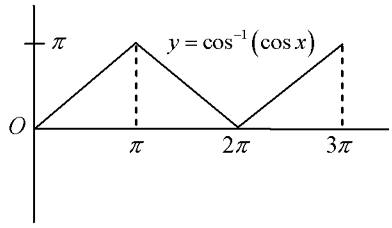
Function	Domain (*D*)	Range (*R*)
$\sin^{-1} x$	$-1 \le x \le 1$ or $[-1, 1]$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$-1 \le x \le 1$ or $[-1, 1]$	$0 \le \theta \le \pi$ or $[0, \pi]$
$\tan^{-1} x$	$x \in R$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ or $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	$x \in R$	$0 < \theta < \pi$ or $(0, \pi)$
$\sec^{-1} x$	$x \le -1$, or $1 \le x$ or $(-\infty, -1] \cup [1, \infty)$	$\theta \ne \frac{\pi}{2}$, $0 \le \theta \le \pi$ or $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

Properties of Inverse Trigonometric Functions

1.(i) $\sin^{-1}(\sin \theta) = \theta$ if and only if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and

$$\sin^{-1}(\sin x) = \begin{cases} x, & 0 \le x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x \le \frac{3\pi}{2} \\ x - 2\pi, & \frac{3\pi}{2} < x \le 2\pi \end{cases}$$



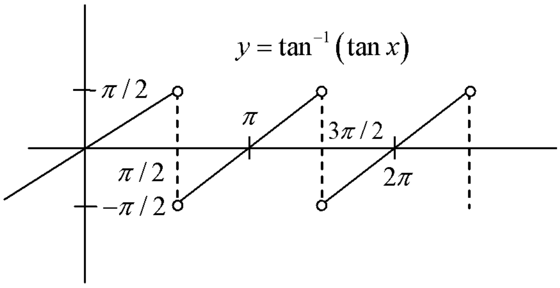


⇒ $f(x) = \sin^{-1}(\sin x)$ is periodic with period 2π.

(ii) $\cos^{-1}(\cos\theta) = \theta$ if and only if $0 \le \theta \le \pi$

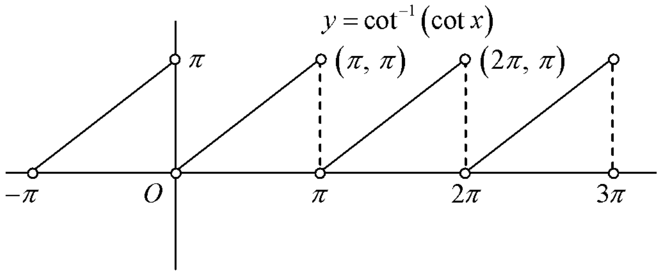
$$\cos^{-1}(\cos x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi < x \le 2\pi \end{cases}$$

⇒ $f(x) = \cos^{-1}(\cos x)$ is periodic with period 2π.



(iii) $\tan^{-1}(\tan\theta) = \theta$ if and only if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and

$$\tan^{-1}(\tan x) = \begin{cases} x, & 0 \le x \le \frac{\pi}{2} \\ x - \pi, & \frac{\pi}{2} < x \le \frac{3\pi}{2} \\ x - 2\pi, & \frac{3\pi}{2} < x \le 2\pi \end{cases}$$

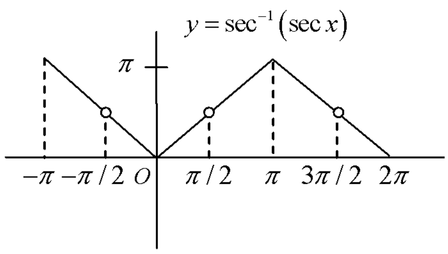


⇒ $f(x) = \tan^{-1}(\tan x)$ is periodic with period π.

(iv) $\cot^{-1}(\cot\theta) = \theta$ if and only if $0 < \theta < \pi$ and

$$\cot^{-1}(\cot x) = \begin{cases} x + \pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \\ x - \pi, & \pi < x < 2\pi \end{cases}$$

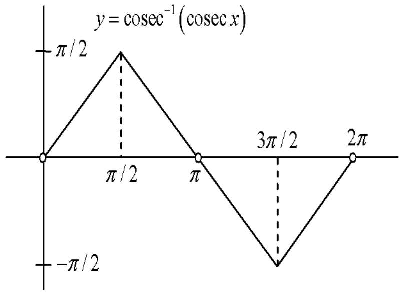
⇒ $f(x) = \cot^{-1}(\cot x)$ is periodic with period π.



(v) $\sec^{-1}(\sec\theta) = \theta$ if and only if $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$ and

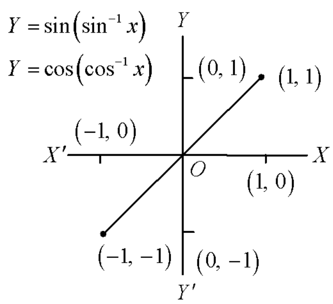
$$\sec^{-1}(\sec x) = \begin{cases} -x, & -\pi \le x \le 0, \quad x \ne -\frac{\pi}{2} \\ x, & 0 < x \le \pi, \quad x \ne \frac{\pi}{2} \end{cases}$$

⇒ $f(x) = \sec^{-1}(\sec x)$ is periodic with period 2π.



(vi) $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta$ if and only if $-\frac{\pi}{2} \le \theta < 0$ or $0 < \theta \le \frac{\pi}{2}$ and

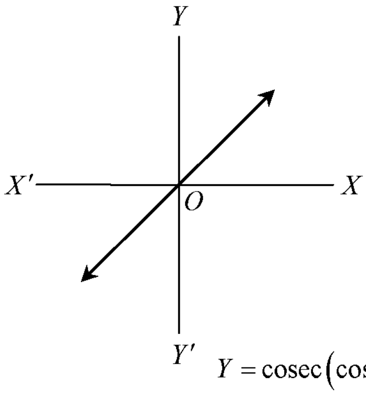
$$\operatorname{cosec}^{-1}(\operatorname{cosec} x) = \begin{cases} x, & 0 < x \le \frac{\pi}{2}, \\ \pi - x, & \frac{\pi}{2} < x \le \frac{3\pi}{2} \quad x \ne \pi \\ x - 2\pi, & \frac{3\pi}{2} < x < 2\pi \end{cases}$$



⇒ $f(x) = \operatorname{cosec}^{-1}(\operatorname{cosec} x)$ is periodic with period 2π.

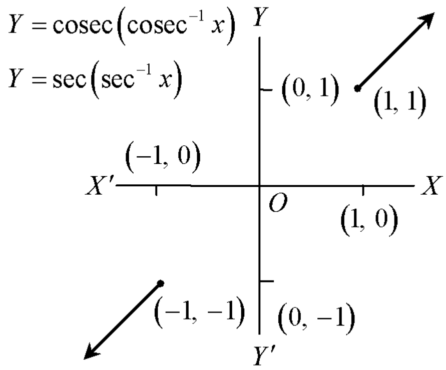
(vii) (a) $\sin(\sin^{-1}x) = x$ iff $-1 \le x \le 1$

(b) $\cos(\cos^{-1}x) = x$ iff $-1 \le x \le 1$



(viii) (a) $\tan(\tan^{-1}x) = x$ for all x

(b) $\cot(\tan^{-1}x) = x$ for all x



(ix) (a) $\sec(\sec^{-1}x) = x$ iff $x \ge 1$ or $x \le -1$

(b) $\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x$ iff $x \ge 1$ or $x \le -1$

2. (i) $\sin^{-1}(-x) = -\sin^{-1}x$, $\cos^{-1}(-x) = \pi - \cos^{-1}x$

(ii) $\tan^{-1}(-x) = -\tan^{-1}x$, $\cot^{-2}(-x) = \pi - \cot^{-1}x$

(iii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$ $\sec^{-1}(-x) = \pi - \sec^{-1}x$,

3. $$\begin{cases} \sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}, & \text{for all } x \in [-1, 1] \\ \tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2}, & \text{for all } x \in R \\ \sec^{-1}x + \operatorname{cosec}^{-1}x = \dfrac{\pi}{2}, & \text{for all } x \in (-\infty, -1] \cup [1, \infty) \end{cases}$$

4. **Principal values for inverse circular functions.**

$$\begin{cases} \sin^{-1}x+\cos^{-1}x=\dfrac{\pi}{2}, & \text{for all } x\in[-1,1] \\ \tan^{-1}x+\cot^{-1}x=\dfrac{\pi}{2}, & \text{for all } x\in R \\ \sec^{-1}x+\operatorname{cosec}^{-1}x=\dfrac{\pi}{2}, & \text{for all } x\in(-\infty,-1]\cup[1,\infty) \end{cases}$$

5. **Conversion property**

(i) $$\begin{cases} \sin^{-1}x = \cos^{-1}\sqrt{1-x^2}, & 0\le x\le 1 \\ \sin^{-1}x = -\cos^{-1}\sqrt{1-x^2}, & -1\le x\le 0 \end{cases}$$

(ii) $$\begin{cases} \sin^{-1}x = \cot^{-1}\dfrac{\sqrt{1-x^2}}{x}, & 0<x\le 1 \\ \sin^{-1}x = \cot^{-1}\dfrac{\sqrt{1-x^2}}{x}-\pi, & -1\le x<0 \end{cases}$$

(iii) $\sin^{-1}x = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$ $|x| < 1$

(iv) $$\begin{cases} \cos^{-1}x = \sin^{-1}\sqrt{1-x^2}, & 0\le x\le 1 \\ \cos^{-1}x = \pi-\sin^{-1}\sqrt{1-x^2}, & -1\le x\le 0 \end{cases}$$

(v) $$\begin{cases} \cos^{-1}x = \tan^{-1}\dfrac{\sqrt{1-x^2}}{x}, & 0<x\le 1 \\ \cos^{-1}x = \pi+\tan^{-1}\dfrac{\sqrt{1-x^2}}{x}, & -1\le x<0 \end{cases}$$

(vi) $\cos^{-1}x = \cot^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$ $|x| < 1$

(vii) $$\begin{cases} \tan^{-1}x = \cos^{-1}\dfrac{1}{\sqrt{1+x^2}}, & x\ge 0 \\ \tan^{-1}x = -\cos^{-1}\dfrac{1}{\sqrt{1+x^2}}, & x\le 0 \end{cases}$$

(viii) $\begin{cases} \tan^{-1} x = \cot^{-1}\frac{1}{x}, & x > 0 \\ \tan^{-1} x = \cot^{-1}\frac{1}{x} - \pi, & x < 0 \end{cases}$

(ix) $\tan^{-1}x = \sin^{-1}\frac{x}{\sqrt{1+x^2}} \quad \forall\ x \in R$

(x) $\begin{cases} \cot^{-1} x = \sin^{-1}\frac{1}{\sqrt{1+x^2}}, & x \geq 0 \\ \cot^{-1} x = \pi - \sin^{-1}\frac{1}{\sqrt{1+x^2}}, & x < 0 \end{cases}$

(xi) $\begin{cases} \cot^{-1} x = \tan^{-1}\frac{1}{x}, & x > 0 \\ \cot^{-1} x = \pi + \tan^{-1}\frac{1}{x}, & x < 0 \end{cases}$

(xii) $\cot^{-1} x = \cos^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) \quad \forall x \in R$

6. **General values of inverse circular functions :** We know that if α is the smallest angle whose sine is *x*, then all the angles whose sine is *x* can be written as $n\pi + (-1)^n \alpha$, where $n \in I$. Therefore, the general value of $\sin^{-1}x$ can be taken as $n\pi + (-1)^n \alpha$.

Thus, we have $\sin^{-1}x = n\pi + (-1)^n \alpha,\ -1 \leq x \leq 1$ if $\sin\alpha = x$ and $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$.

Similarly, general values of other inverse circular functions are given as follows :

$\cos^{-1}x = 2n\pi \pm \alpha,\ -1 \leq x \leq 1$; If $\cos\alpha = x,\ 0 \leq \alpha \leq \pi$

$\tan^{-1}x = n\pi + \alpha,\ x \in R$; If $\tan\alpha = x,\ -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$

$\cot^{-1}x = n\pi + \alpha,\ x \in R$; If $\cot\alpha = x,\ 0 < \alpha < \pi$

$\sec^{-1}x = 2n\pi \pm \alpha,\ x \leq -1$ or $x \geq 1$ If $\sec\alpha = x,\ 0 \leq \alpha \leq \pi$ and $\alpha \neq \frac{\pi}{2}$

Note :

The first letter in all above inverse Trigonometric function are CAPITAL LETTER.

Formulae for sum, difference of inverse trigonometric function

(1) $\begin{cases} \sin^{-1} x + \sin^{-1} y = \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}; & x \geq 0,\ y \geq 0 \text{ and } x^2 + y^2 \leq 1 \\ \sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}; & x \geq 0,\ y \geq 0 \text{ and } x^2 + y^2 \geq 1 \end{cases}$

(2) $\sin^{-1}x - \sin^{-1} y = \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}$; $x \ge 0,\ y \ge 0$

(3) $\cos^{-1}x + \cos^{-1} y = \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\}$; $x \ge 0,\ y \ge 0$

(4) $$\begin{cases} \cos^{-1}x - \cos^{-1}y = \cos^{-1}\left\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\right\}; & x \ge 0,\ y \ge 0,\ x \le y \\ \cos^{-1}x - \cos^{-1}y = -\cos^{-1}\left\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\right\}; & x \ge 0,\ y \ge 0,\ x \ge y \end{cases}$$

(5) $$\begin{cases} \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right); & x \ge 0,\ y \ge 0 \text{ and } xy < 1 \\ \tan^{-1}x + \tan^{-1}y = \pi/2\ ; & x > 0,\ y > 0 \text{ and } xy = 1 \\ \tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right); & x \ge 0,\ y \ge 0 \text{ and } xy > 1 \end{cases}$$

(6) $\tan^{-1}x - \tan^{-1} y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$; $x \ge 0,\ y \ge 0$

Inverse trigonometric ratios of multiple angles

1. $$\begin{cases} 2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right) & \text{If } -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ 2\sin^{-1}x = \pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right) & \text{If } \dfrac{1}{\sqrt{2}} < x \le 1 \\ 2\sin^{-1}x = -\pi + \sin^{-1}\left(2x\sqrt{1-x^2}\right) & \text{If } -1 \le x \le \dfrac{-1}{\sqrt{2}} \end{cases}$$

2. $$\begin{cases} 3\sin^{-1}x = \sin^{-1}\left(3x-4x^3\right), & \text{If } \dfrac{-1}{2} \le x \le \dfrac{1}{2} \\ 3\sin^{-1}x = \pi - \sin^{-1}\left(3x-4x^3\right), & \text{If } \dfrac{1}{2} < x \le 1 \\ 3\sin^{-1}x = -\pi - \sin^{-1}\left(3x-4x^3\right), & \text{If } -1 \le x < -\dfrac{1}{2} \end{cases}$$

3. $$\begin{cases} 2\cos^{-1}x = \cos^{-1}\left(2x^2-1\right), & \text{If } 0 \le x \le 1 \\ 2\cos^{-1}x = 2\pi - \cos^{-1}\left(2x^2-1\right), & \text{If } -1 \le x \le 0 \end{cases}$$

4. $$\begin{cases} 3\cos^{-1} x = \cos^{-1}\left(4x^3 - 3x\right), & \text{If } \frac{1}{2} \le x \le 1 \\ 3\cos^{-1} x = 2\pi - \cos^{-1}\left(4x^2 - 3x\right), & \text{If } -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1} x = 2\pi + \cos^{-1}\left(4x^3 - 3x\right), & \text{If } -1 \le x \le -\frac{1}{2} \end{cases}$$

5. $$\begin{cases} 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{If } -1 < x < 1 \\ 2\tan^{-1} x = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{If } x > 1 \\ 2\tan^{-1} x = -\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{If } x < -1 \end{cases}$$

6. $$\begin{cases} 2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{If } -1 \le x \le 1 \\ 2\tan^{-1} x = \pi + \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{If } x > 1 \\ 2\tan^{-1} x = -\pi + \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{If } x < -1 \end{cases}$$

7. $$\begin{cases} 2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), & \text{If } 0 \le x \\ 2\tan^{-1} x = -\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), & \text{If } x \le 0 \end{cases}$$

8. $$\begin{cases} 3\tan^{-1} x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), & \text{If } \frac{-1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ 3\tan^{-1} x = \pi + \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), & \text{If } x > \frac{1}{\sqrt{3}} \\ 3\tan^{-1} x = -\pi + \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), & \text{If } x < -\frac{1}{\sqrt{3}} \end{cases}$$